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# OPTIMAL DYNAMIC INVESTMENT POLICY UNDER FINANCIAL RESTRICTIONS AND ADJUSTMENT COSTS

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In this paper we extend the well known dynamic adjustment cost models [Gould (1968), Nickell (1978), Treadway (1969)] by incorporating financing restrictions. It turns out that the firm has to invest at its maximum before it reaches the optimal situation derived from the results of those models. Some other conclusions are that investment is a continuous function of time, that it is not optimal for the firm to invest at the end of the planning period when the planning horizon is finite and that a decision rule can be derived to determine uniquely the firm's optimal investment policy. This new dynamic approach leads to a simple decision procedure in practical situations.

## 1. Introduction

Empirical studies have shown that the development of the firm over time can be divided into different stages, such as growth, stationarity and contraction. In order to understand, evaluate and control these stages, economists have used dynamic mathematical techniques like optimal control theory, calculus of variations and dynamic programming to develop and analyse dynamic models of the firm.

One of the first dynamic models of the firm is the classical model of Jorgenson (1963) with an optimal solution that dictates an instantaneous adjustment of the stock of capital goods to the level of maximum revenue. Later scientists introduce two ways in order to avoid this unrealistic immediate adjustment, namely by the introduction of financing limits [e.g., Leland (1972), Ludwig (1978) and Van Loon (1983)] and by incorporating adjustment costs [e.g., Gould (1968), Nickell (1978) and Treadway (1969)]. Van Schijndel (1986, 1987) extended the financial models by introducing taxation.

In the literature, however, no attention has been paid to a combination of these two topics. In this paper, therefore, we will analyse the impact of both

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financing limits and adjustment costs on the firm's optimal dividend/ investment policy in a dynamic setting.

## 2. Model formulation and solution concept

First, we formulate the model in symbols; afterwards the definition of the symbols and the interpretation of the model will be given.

$$\underset{I,D}{\text{maximize}} \quad \int_0^z D(T)\,\mathrm{e}^{-iT}\,\mathrm{d}T + X(z)\,\mathrm{e}^{-iz} \tag{1}$$

$$\text{subject to} \quad \dot{K} = I(T) - aK(T), \qquad K(0) = K_0 > 0, \tag{2}$$

$$\dot{X} = S(K) - aK(T) - U(I) - D(T), \qquad X(0) = X_0 > 0, \tag{3}$$

$$K(T) = X(T), \tag{4}$$

$$D(T) \geqq 0, \tag{5}$$

$$I(T) \geqq 0, \tag{6}$$

with $D$ = dividend, $I$ = gross investment, $K$ = capital goods, $S$ = earnings, $S(K) > 0$, $\mathrm{d}S/\mathrm{d}K > 0$, $\mathrm{d}^2S/\mathrm{d}K^2 < 0$, $T$ = time, $U$ = adjustment costs, $U(I) \geqq 0$, $\mathrm{d}U/\mathrm{d}I > 0$, $\mathrm{d}^2U/\mathrm{d}I^2 > 0$, $U(0) = 0$, $X$ = equity, $a$ = depreciation rate, $i$ = discount rate and $z$ = planning horizon.

In order to avoid an unrealistic solution, we assume that operating income must be positive:

$$O(K) = S(K) - aK - U(aK) > 0. \tag{7}$$

The model describes a firm that maximizes its value to the shareholders (1). The state of the firm is fixed by the values of its equity (2) and capital good stock (3). The firm finances its assets by equity (4), it operates under decreasing returns to scale and under a convex adjustment cost function.

Due to (2), (3) and the derivative of (4), we can derive

$$D = S(K) - I - U(I). \tag{8}$$

The above substitution enables us to get rid of the expressions (3), (4) and (5) by adding

$$S(K) - I - U(I) \geqq 0. \tag{9}$$

Using standard control theory application [see Feichtinger and Hartl (1986)], we derive the Lagrange function which includes the Hamiltonian and the constraints

$$L=(S(K)-I-U(I))\,\mathrm{e}^{-iT}+\psi(I-aK)+\lambda_1(S(K)-I-U(I))+\lambda_2 I, \tag{10}$$

then the necessary conditions are

$$\frac{\partial L}{\partial I}=-\left(1+\frac{\mathrm{d}U}{\mathrm{d}I}\right)(\mathrm{e}^{-iT}+\lambda_1)+\psi+\lambda_2=0, \tag{11}$$

$$-\dot{\psi}=\frac{\mathrm{d}S}{\mathrm{d}K}(\mathrm{e}^{-iT}+\lambda_1)-a\psi, \tag{12}$$

$$\lambda_1(S(K)-I-U(I))=0, \tag{13}$$

$$\lambda_2 I=0, \tag{14}$$

$$\psi(z)=\mathrm{e}^{-iz}. \tag{15}$$

As the Hamiltonian is concave in $(K, I)$ and the functions $S(K)-I-U(I)$ and $I$ are quasi-concave in $(K, I)$, these conditions are also sufficient [see Van Loon (1983, p. 105)].

Next, we apply Van Loon's general solution procedure [see Van Loon (1983, p. 116)] in order to transform these conditions into the optimal trajectories of the firm. Each trajectory consists of a succession of feasible paths, which are each of them characterized by the set of active constraints. The properties of these paths are presented by table 1.

Table 2 gives us a survey of the feasible and infeasible couplings. The formal proofs of the results presented in this table can be obtained from the author upon request.

From table 2 we derive the following possible trajectories:

path 1 – path 2 – path 3
path 2 – path 3
path 3 – path 2 – path 3
path 3

Depending on $K_0$ and the length of the planning period one of these trajectories is optimal. In the next section we analyse the first mentioned one.

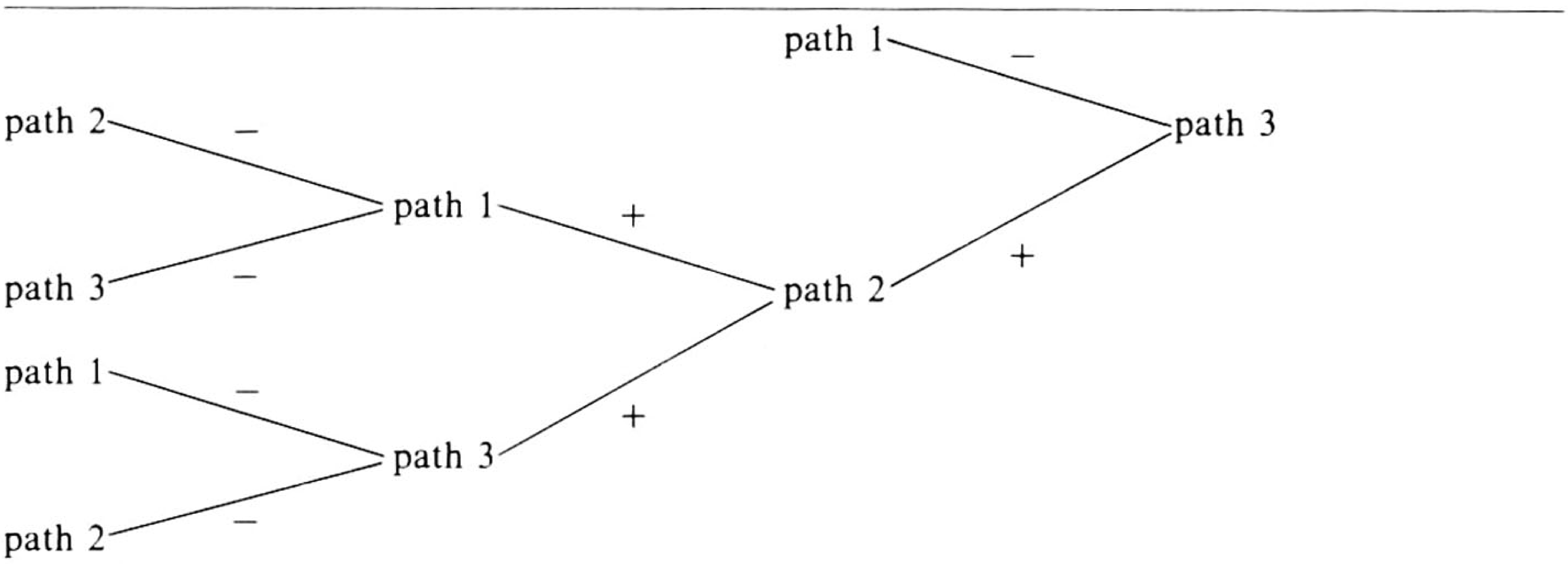
## 3. Economic analysis

In this section we discuss the first one of the four feasible trajectories that

Table 1

| Path | $\lambda_1$ | $\lambda_2$ |
|---|---|---|
| 1 | + | 0 |
| 2 | 0 | 0 |
| 3 | 0 | + |
| 4 | + | + |

Table 2[a]




[a] +: coupling is feasible,
−: coupling is infeasible.

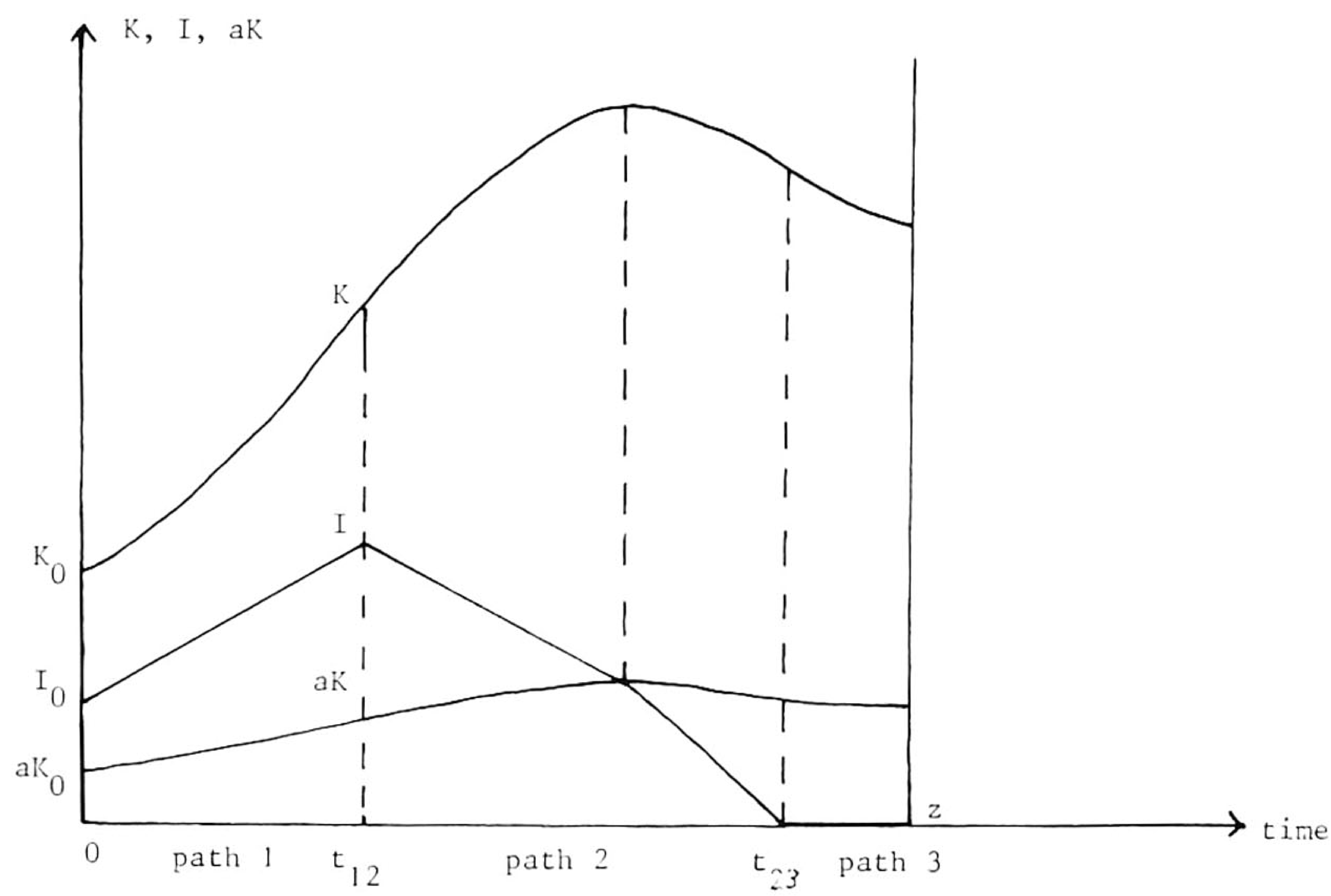



Fig. 1. Development of $K$, $I$, and $aK$ during the planning period.

are mentioned in the previous section. This trajectory, which is depicted by fig. 1, may be considered as a master trajectory, because the second and the fourth are subsections of it. The third mentioned pattern arises only when the initial value $K_0$ is very large. Because of the low marginal earnings at these levels, the optimal policy is to start with investment equal to zero.

Concerning fig. 1 we in advance remark that the way $I$ rises on path 1 and diminishes on path 2 depends completely on specific features of $S(K)$ and $U(I)$. So, since $S(K)$ and $U(I)$ are not specified, we do not know whether the slope of $I$ increases, decreases or remains constant on these paths.

On path 1 the firm invests at its maximum level, i.e., the level which is feasible subject to the financing restriction. Consequently the firm does not pay out any dividend and invests all earnings. This policy is optimal, because marginal earnings of investment exceed marginal cost, which is shown by the next expression:

$$1+\frac{\mathrm{d}U}{\mathrm{d}I}<\mathrm{e}^{-(i+a)(z-T)}+\int_T^z \mathrm{e}^{-(i+a)(t-T)}\frac{\mathrm{d}S}{\mathrm{d}K}(t)\,\mathrm{d}t. \tag{16}$$

The left-hand side of this relation represents an investment expenditure of one dollar including adjustment costs, whereas the right-hand side expresses the marginal earnings of investments which consists of the present value of the remaining new equipment at the end of the planning period (the value of the new equipment decreases with depreciation rate $a$ during the rest of the planning period) plus the present value of additional sales over the whole period due to this new equipment (the production capacity of this equipment decreases with rate $a$ during the rest of the planning period).

At $t_{12}$ the firm stops with growing at its maximum, because otherwise marginal earnings would become too small ($\mathrm{d}S/\mathrm{d}K$ decreases when $K$ rises) to finance the rising marginal adjustment costs ($\mathrm{d}U/\mathrm{d}I$ rises when $I$ rises). Therefore, on path 2 investment is kept on such a level that marginal earnings equal marginal cost. Expression (16) therefore changes into

$$1+\frac{\mathrm{d}U}{\mathrm{d}I}=\mathrm{e}^{-(i+a)(z-T)}+\int_T^z \mathrm{e}^{-(i+a)(t-T)}\frac{\mathrm{d}S}{\mathrm{d}K}(t)\,\mathrm{d}t, \tag{17}$$

From this expression we conclude that the net present value of the last investment unit equals zero, which means that investment is at its optimal level during path 2. On path 1 this situation cannot be reached because of the active financing restriction. At the beginning of path 2 investment decreases, but capital stock still increases until $I$ falls below the depreciation level. From this very moment $K$ will also drop.

Just when investment becomes zero, path 2 passes into path 3. This transition is fixed by the moment that the next expression becomes applicable:

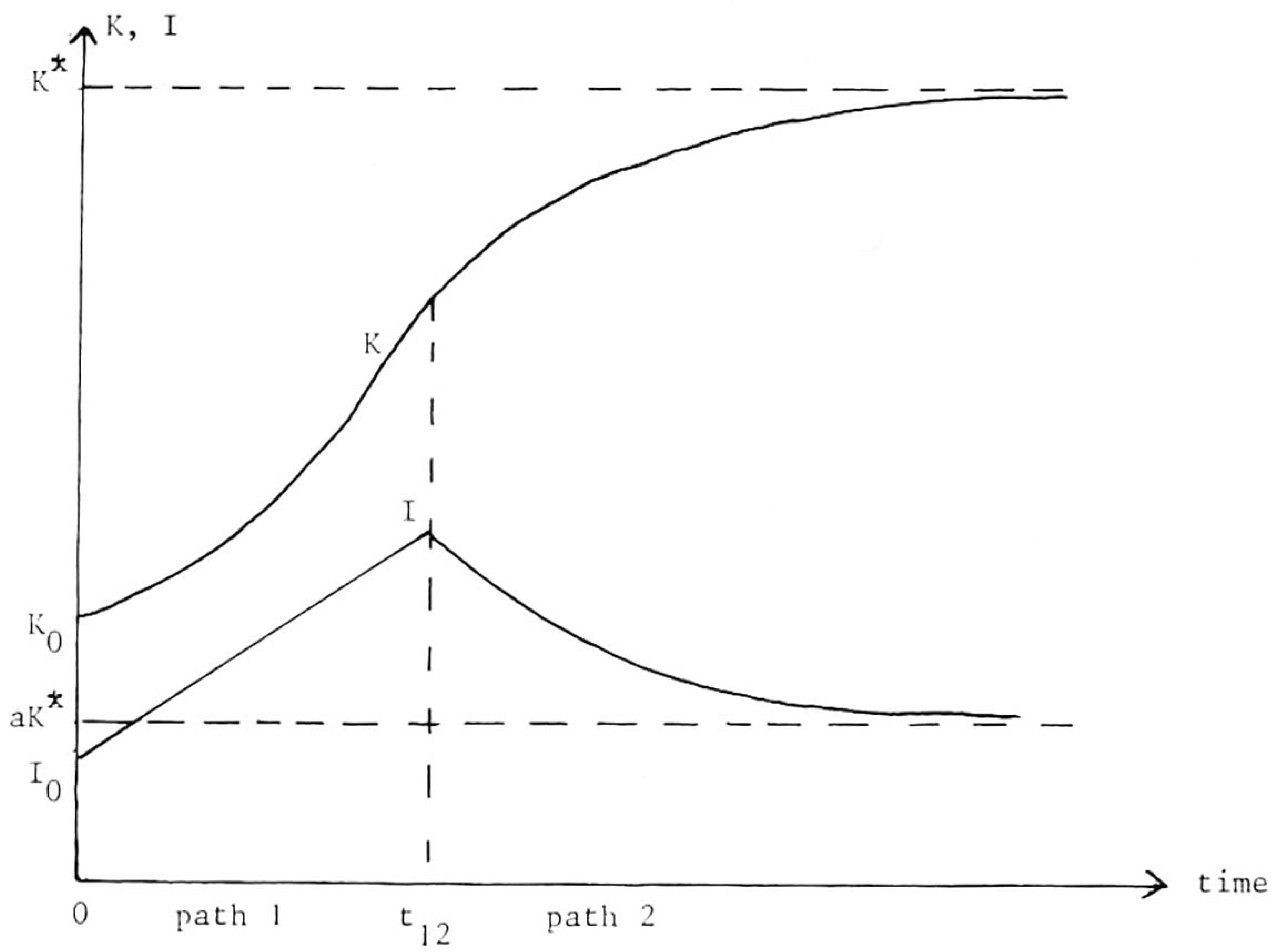



Fig. 2. Development of $K$ and $I$ during the time in the case of an infinite time horizon.

$$1+\frac{\mathrm{d}U}{\mathrm{d}I}>\mathrm{e}^{-(i+a)(z-T)}+\int_T^z \mathrm{e}^{-(i+a)(t-T)}\frac{\mathrm{d}S}{\mathrm{d}K}(t)\,\mathrm{d}t. \tag{18}$$

The inequality shows us that marginal cost of investment exceeds marginal earnings on path 3. This is caused by the fact that from $t_{23}$ on the remaining time period is too short to defray the adjustment costs of new investments. Therefore, the firm does not invest anymore on path 3. The formal proofs of the relations (16), (17) and (18) can be obtained from the author on request.

A big difference between this model and the usual dynamic financial models of the firm [e.g., Leland (1972), Ludwig (1978) and Van Loon (1983)] is the fact that now investment is a continuous variable during the time. The mathematical reason for this is, that the control function has to be continuous within a regular optimal control problem [see Feichtinger and Hartl (1986, p. 168)]. From economical point of view it can be argued that a gradual development of $I$ is optimal, because large investment expenditures imply very high adjustment costs.

Another interesting feature is the way in which this pattern will change in case the planning period is extended. If $z$ is fixed upon a higher value the firm has more time to grow, so $t_{12}$ and $t_{23}$ will be postponed. In the case of an infinite time horizon, expression (17) continues to hold from $t_{12}$ on, which implies that path 2 does not pass into path 3 anymore. This is easy to understand, because in the case of an infinite time horizon there is always enough time to defray the adjustment costs. On path 2 $K$ will approach a stationary value asymptotically (fig. 2). Here, the influence of the convex

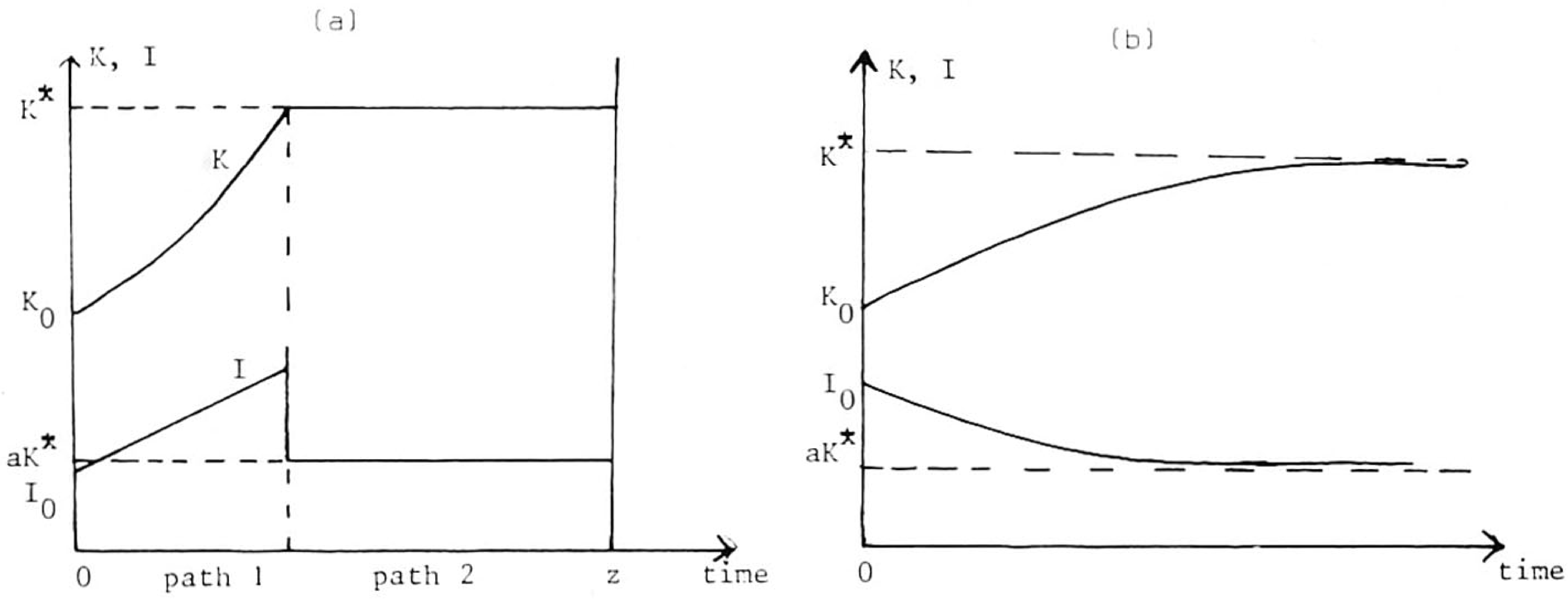



Fig. 3. Development of $K$ and $I$ during the time, in case of an investment model with financing limits (a) and with adjustment costs (b).

adjustment cost function becomes clear; the stationary value will never be reached (in contrast with usual dynamic models), because it is always cheaper to split up the final adjustment into two parts.

At the end of this article we will compare the solution of our model to solutions that arise from a model which has incorporated only financing limits and models which contain only adjustment costs.

The optimal trajectory of a financial model without adjustment costs exists of two paths. On path 1 the firm will grow as fast as possible until an equalization of marginal costs $(i+a)$ and marginal earnings $(\mathrm{d}S/\mathrm{d}K)$ is reached. Then, on path 2 investment falls down to replacement level, which implies that capital remains constant.

The above is represented in fig. 3a. In this solution the level of $K^*$ will exceed the maximum level of $K$ in the financial model with adjustment costs. Also, the firm will grow faster in this model, because growth is only limited here by the financing restriction, whereas in the financial adjustment cost model growth is additionally restricted by the adjustment costs.

Several authors, such as Gould (1968), Treadway (1969) and Nickell (1978) have studied adjustment cost models without financing limits. In these models investment is not restricted by an upperbound. Another common feature is that the above authors all suppose an infinite time horizon, so investment will never be at its lower bound either. Therefore, it is not surprising that the solutions of these kinds of models only contain either eq. (17) [Nickell (1978)], or the derivative of (17) to time [Gould (1968) and Treadway (1969)], for eq. (17) holds on path 2 of our model and the main characteristic of this path is that investment is not one of its bounds. The solution of such an adjustment cost model is represented in fig. 3b.

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